Climb of a bore on a beach
Part 3. Run-up

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When a bore travels shoreward into water at rest on a beach, then according to the first-order non-linear long-wave theory, the bore accelerates and decreases in height, until it collapses at the shore. The investigation here reported concerns the question, what happens next? It is formulated as a singular characteristic boundary-value problem with somewhat unusual mathematical properties. Its asymptotic solution predicts a rather thin sheet of run-up and back-wash with some unexpected features.

1. Introduction

Parts 1 (Ho & Meyer 1962) and 2 (Shen & Meyer 1963) are concerned with the mathematical question of the 'memory' of a class of solutions of a non-linear hyperbolic problem, and hydrodynamical language is used only to conform to convention. However, being indebted to Dr Van Dorn of the Scripps Institution of Oceanography for impressing upon us the practical importance of the age-old problem of surf—breakers, run-up and back-wash—we now apply the non-linear shallow-water theory speculatively to waves on a beach.

The situation envisaged corresponds to long swell for which the shore behaviour of each wave is virtually independent of the back-wash of the preceding wave and each travels effectively into water at rest. Since the gasdynamical analogy (Stoker 1957) furnishes a fair degree of understanding of the process of bore formation, the present investigation begins only when the bore is well developed and forms the head of the wave (figure 1). The bore is considered as a discontinuity of water velocity and surface elevation such that mass and horizontal momentum are conserved [(I, 4) to (I, 6)]† The continuous water motion is considered to satisfy the first-order shallow-water equations (I, 2), (I, 3). As in part 1, and in contrast to part 2, only beaches of uniform slope will be considered.

Part 1 discusses the development of the bore up to the time $t = 0$ when it reaches the original shore position $x = 0$. Since the equations are hyperbolic, this involves only a partial analysis of the water motion. It is best to refer to an (x, t)-diagram (figure 1). If the successive bore positions are represented by the 'bore path' B, and the 'limiting characteristic' (§ I, 3) by the curve L, then analysis of the water motion at the times and positions represented by points in the shaded region (figure 1) between L and B suffices for the determination of the

† Equations and section numbers of part 1 (Ho & Meyer 1962) will be distinguished by a suffix I.

bore development for $t \leqslant 0$. In particular, the shape and velocity distribution of the waves to seaward of the bore need be known at the initial time T only for $X_0 \leqslant x \leqslant X$ (figure 1).

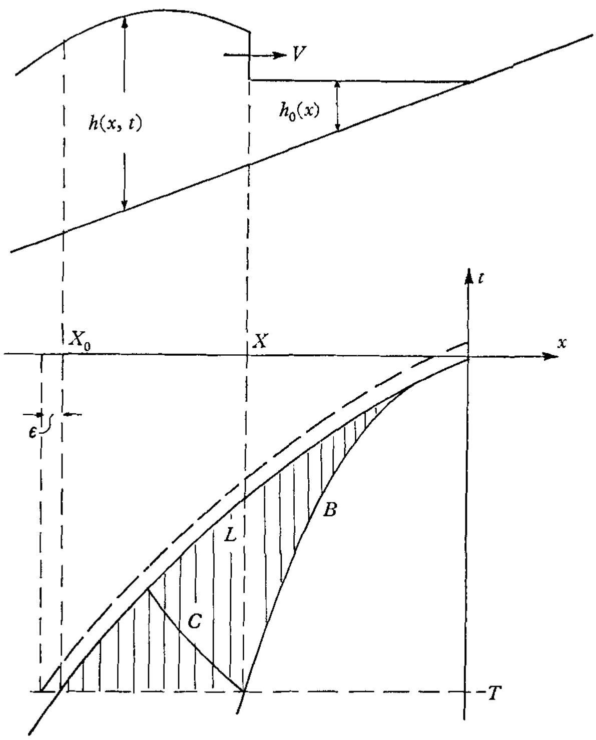

FIGURE 1. Definition sketch and (x, t)-diagram showing locus B of successive bore positions, 'limiting characteristic' L and 'seaward boundary characteristic' C. A greatly contracted horizontal scale is implied.

Actually, the initial conditions on $t = T$ are replaced in part 1 by an equivalent boundary condition on the segment between B and L of the receding characteristic line C passing through (X, T) (figure 1), referred to as 'seaward boundary'. The boundary condition adopted involves no detailed specification of wave shape, but only a monotoneity assumption or inequality for the acceleration (§I, 4; its *a priori* justification is not here attempted any more than that of the governing equations). The asymptotic bore behaviour, as $t \to 0$, is then deduced in part 1 without any further assumptions or approximations. The main result is that a singularity of the water acceleration occurs, even though the bore height tends to zero and the bore velocity, like the water velocity just to seaward of the bore, tends to a positive limit u_0.

The investigation to be reported now concerns the further development of the water motion, and in particular, the motion of the shore line defined by $h(x, t) = 0$ (figure 1), for $t > 0$. Such analysis requires a knowledge of the initial wave shape and velocity distribution to seaward beyond $x = X_0$ (figure 1), but it will be assumed only that the initial data are extended over an arbitrarily short distance $X_0 - \epsilon \leqslant x \leqslant X_0$. More specifically, it will be assumed that the monotoneity assumption of §I, 4 remains valid on C (figure 1) for an arbitrarily short time longer than is required for the results of part 1. We may then expect the solution to be determined asymptotically in the narrow strip between L and a neighbouring characteristic line of the same family, shown dashed in figure 1. The shore line movement should thus be determined for $0 \leqslant t \leqslant t_0 = O(\epsilon)$, and as $\epsilon \to 0$, we may expect to find the initial velocity of any reflected bore and the initial velocity of the shore line.

The expectation is not at all borne out, however, when a bore heads the incoming wave. The little singularity of the acceleration for $t \uparrow 0$ discovered in part 1 will be shown to be the herald of a magnificent singularity for $t \downarrow 0$, which yields the shore line movement for $t > 0$ over a time interval independent of ϵ, in fact, over the duration of the whole run-up and part of the backwash! It is in the nature of a singularity, moreover, that it dominates the solution over a whole region, and thus firm indications concerning the internal structure of run-up and back-wash under normal circumstances will be obtained, even though nothing but the very shore line is strictly determined in the limit $\epsilon = 0$. Despite the hyperbolic character of the governing equations, therefore, much of the run-up and back-wash turns out to be determined by the same, restricted portion $X_0 \leqslant x \leqslant X$ (figure 1) of the incident wave which determines the shoreward travel of the bore.†

Nor is a detailed, quantitative knowledge of even that portion of the incident wave required. Its only quantitative property possessing a major influence on the observable predictions is the basic velocity scale u_0, just as in parts 1 and 2. An acceleration parameter a_0 representing the wave plays again a crucial role in the mathematical argument (§2 below), but only its signature matters; its actual value should, at best, be barely observable.

Three physical predictions will be derived (§3). The shore line has discontinuous velocity and acceleration. Both vanish for all $t < 0$. But as $t \downarrow 0$, the shore line velocity tends to $u_0 > 0$, and its acceleration, to $g\,dh_0/dx < 0$, and the shore line maintains this constant deceleration during the whole run-up and part of the back-wash. This stands in contrast to the particular solutions of Carrier & Greenspan (1958) in which the shore line performs a smooth oscillation with variable acceleration.

The net water height, close to the shore line, is found proportional to the square of the distance from the shore line, during the run-up and part of the back-wash. This is again in contrast to the particular solutions of Carrier & Greenspan (1958), for which the net water height, close to shore, is generally proportional to the shore distance. At a fixed, small distance from the (moving) shore line, moreover, the net water height is here found proportional to t^{-2}, if t denotes again the time since the start of the run-up, and this also holds for the whole run-up and part of the back-wash. It thus appears that the notable thinness of the sheet of run-up and especially, back-wash, need not be explained entirely from seepage through the sand.

† Apart from its physical implications, this result may be of some mathematical interest. The governing differential equations are hyperbolic, but exhibit a behaviour differing radically from that characteristic of hyperbolic equations. This is due, of course, to the fact that the equations are singular.

The third prediction concerns the formation of, not a reflected, but a curious secondary bore in the interior of the back-wash (§3).

We have limited our attention to qualitative predictions of the model. The quantitative predictions, and also some features of the qualitative ones, must clearly be subject to modification by a variety of physical effects which cannot be described by the conventional shallow-water equations. But it is remarkable that so simple a set of non-linear equations should go as far as it does in describing the whole phenomenon of long surf, from bore-formation to back-wash.

2. Formal solution

To avoid technical complications, it is necessary to concentrate on the canonical variables $a(\alpha,\beta)$ and $b(\alpha,\beta)$ defined by

$$a = (\alpha+\beta)^{\frac{3}{2}}\,\partial t/\partial\alpha, \qquad b = (\alpha+\beta)^{\frac{3}{2}}\,\partial t/\partial\beta, \tag{1}$$

$$\alpha = 2c+u+\gamma t-u_0, \quad \beta = 2c-u-\gamma t+u_0, \tag{2}$$

$$c^2 = gh(x,t), \qquad \gamma = -gh_0/x = \text{const.} > 0, \tag{3}$$

where t denotes the time after arrival of the bore at the initial shore position, x the horizontal distance, measured landward from that position, u the water velocity in the x-direction, h the local water depth, h_0 the equilibrium water depth, g the gravitational acceleration and u_0 the limit of the bore velocity (§I, 3). The two-dimensional motion of water on a beach is then governed, according to the first-order, non-linear shallow-water theory (§I, 2), by

$$\partial a/\partial\beta = -\tfrac{3}{2}(\alpha+\beta)^{-1}b, \qquad \partial b/\partial\alpha = -\tfrac{3}{2}(\alpha+\beta)^{-1}a. \tag{4}$$

Once $a(\alpha,\beta)$ and $b(\alpha,\beta)$ are known, $t(\alpha,\beta)$ and $x(\alpha,\beta)$ are found from (1) and the characteristic equations,

$$\partial x/\partial\beta = (u+c)\,\partial t/\partial\beta, \qquad \partial x/\partial\alpha = (u-c)\,\partial t/\partial\alpha, \tag{5}$$

by quadrature.

The monotoneity and regularity assumptions of §I, 4 may be expressed as

$$\left.\begin{array}{l} a(\alpha,\beta_0) > 0, \\ a \text{ and } \partial a/\partial\alpha \text{ are piecewise continuous for } \beta = \beta_0. \end{array}\right\} \tag{6}$$

In §I, 4, (6) is set for $\alpha \leqslant 0$, and it is now extended to $0 \leqslant \alpha \leqslant \alpha_0$, where α_0 is an arbitrary, positive number; so is β_0, except for an upper bound discussed in §I, 3. We may add the results of §I, 5 that on L, i.e. for $\alpha = 0, \beta \geqslant 0$,

$$\left.\begin{array}{r} a(0,\beta) = a_0+O(c^{\frac{9}{2}}) > 0, \\ b(0,\beta) = O(c^{\frac{9}{2}}) < 0, \\ t(0,\beta) = O(c^4) < 0, \\ x(0,\beta) = O(c^4) < 0. \end{array}\right\} \tag{7}$$

Equations (4), (6) and (7) formulate a characteristic boundary-value problem for the region G_0 defined by $0 \leqslant \alpha \leqslant \alpha_0$, $0 \leqslant \beta \leqslant \beta_0$ (figure 2), of which a unique and continuous solution is known (Courant 1962) to exist in the region G_δ defined by $0 \leqslant \alpha \leqslant \alpha_0$, $0 < \delta \leqslant \beta \leqslant \beta_0$. There is no certainty that a solution exists in a straightforward sense in G_0, since (4) are singular at $\alpha+\beta = 0$, and our first preoccupation must be with letting $\delta \to 0$.

Observe that the problem just defined is a formal one, since we have transformed away the non-linearity of the shallow-water equations (§I, 2) without regard to the implications of such a procedure. Our second preoccupation must therefore be with the question whether the solution is physically admissible, and in particular, whether it predicts single-valued functions $\alpha(x,t)$ and $\beta(x,t)$ in the regions corresponding to G_δ and G_0.

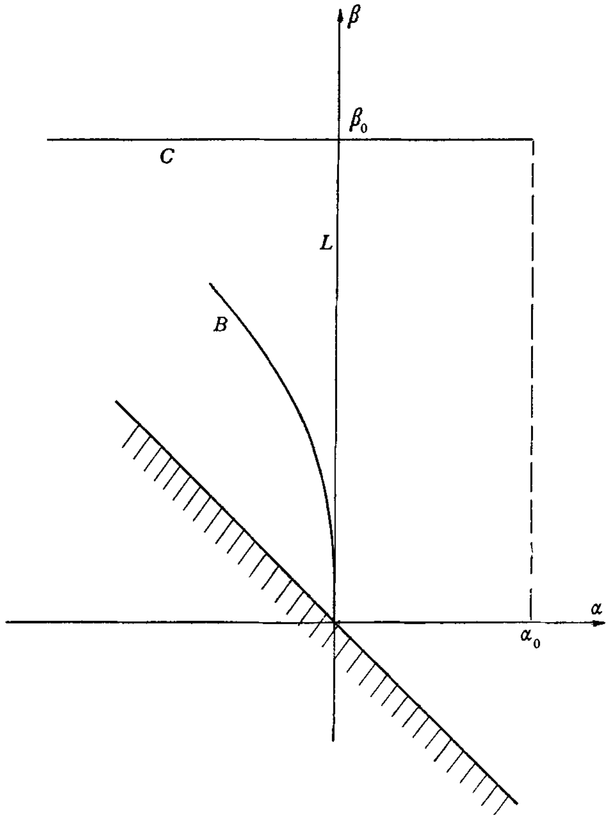

FIGURE 2. Diagram of the characteristic plane showing bore path B, limiting characteristic L and seaward boundary characteristic C.

The technical discussion of the solution is deferred to the Appendix, where we prove

Lemma 1. Let $\alpha = k\beta$, with $0 \leqslant k \leqslant K$, where K is arbitrary. Then as $\beta \downarrow 0$ with k fixed,

$$a+b \to a_0 F(-\tfrac{3}{2},\tfrac{5}{2};\ 1;\ \alpha/(\alpha+\beta)),$$

$$a-b \to a_0 F(-\tfrac{1}{2},\tfrac{3}{2};\ 1;\ \alpha/(\alpha+\beta)),$$

uniformly in G_0, where F denotes the hypergeometric function.

Lemma 2.

$$\lim_{k\to\infty}\lim_{\beta\to+0} a(k\beta,\beta) = -2a_0/(3\pi),$$

$$\lim_{k\to\infty}\lim_{\beta\to+0} [b(k\beta,\beta)/\log(1+k)] = a_0/\pi$$

then follows readily from known properties of the hypergeometric function (Barnes 1908; Whittaker & Watson 1940).

A comparison of Lemma 2 with (7) shows that both a and b must change sign in G_0. Since a and b are known to be continuous in G_δ, and Lemma 1 shows them to be continuous also in G_0, except at the boundary $\beta = 0$, such change of sign must occur at 'singular lines' $a = 0$ and $b = 0$. By Lemma 1, these singular lines must approach the origin in directions $\alpha/\beta = k$, with $0 < k < \infty$ and with

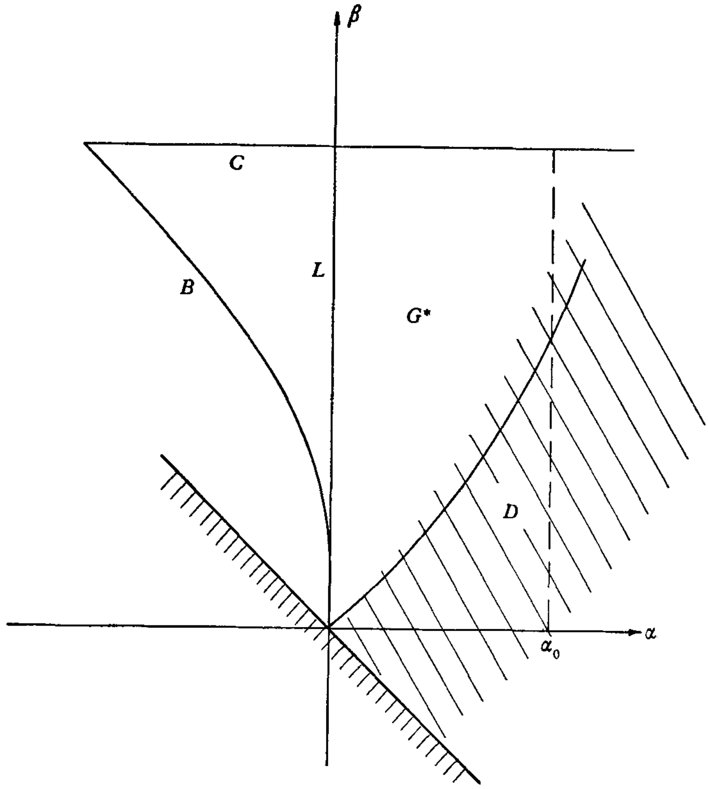

FIGURE 3. Diagram of the characteristic plane showing course of first singular line D schematically.

different values of k for different singular lines. Any intersection of singular lines in G_0 can therefore be avoided, if α_0 be chosen sufficiently small. It is known (Meyer 1949) that $\alpha(x, t)$, $\beta(x, t)$ are not single-valued at the 'limit lines' which are the images of the singular lines in the (x, t)-plane, and the singular lines therefore represent a bound to the extent of the (α, β)-region in which the formal solution can be admissible. Thus if D denotes the singular line arrived at first, as α increases from zero, for fixed β (figure 3), then the region G^* of the (α, β)-plane in which the formal solution is admissible is that between the bore B and the region shaded in figure 3. Observe that, since the mapping into the (x, t)-plane is 1–1 in the sector between B and D (figure 3), a comparison of figures 1 and 3 also shows the limit line corresponding to D to be that first arrived at with increasing time, in the (x, t)-plane.

The precise extent of G^* will not be relevant to what follows, but it will be of interest to know whether $a = 0$ or $b = 0$ on D. The answer is given by an interlacing theorem (Meyer 1949); by (4) and (7), as α increases from zero, for fixed β, a must change sign before b can vanish, and hence, $a = 0$ on D. Now, in so far as limit lines are connected with bore formation (Stoker 1957; Meyer 1960), an 'advancing' limit line $a = 0$ is connected with an 'advancing' bore. Therefore, our result does not indicate the reflexion of the original (advancing) bore of part I as a receding bore, but rather seems to throw doubt on the physical relevance of the formal solution. This can be elucidated only by a study of the mapping $(\alpha, \beta) \to (x, t)$, especially near the extraordinary point $\alpha = \beta = 0$.

3. Interpretation

From (1), (2) and (5),

$$t(\alpha,\beta)-t(0,\beta) = \int_0^\alpha (\alpha'+\beta)^{-\frac{5}{2}}\, a(\alpha',\beta)\, d\alpha', \tag{8}$$

$$x(\alpha,\beta)-x(0,\beta) = \int_0^\alpha (\alpha'+\beta)^{-\frac{5}{2}}\, a(\alpha',\beta)\,[u_0-\gamma t(\alpha',\beta)+(\alpha'-3\beta)/4]\, d\alpha', \tag{9}$$

and by (7), $t(0, \beta) \to 0$ and $x(0,\beta) \to 0$ as $\beta \downarrow 0$. The form of (8), (9) and Lemma 1 shows that an approach to $\alpha = \beta = 0$ along three families of curves

$$\alpha = \mu\beta^\lambda, \quad \lambda = \text{const.}, \quad 0 \leqslant \mu = \text{const.} < \infty$$

should be considered, with a distinction between (i) the 'steep' curves for which $\lambda > \frac{3}{2}$, (ii) the 'main' curves for which $\lambda = \frac{3}{2}$, and (iii) the 'flat' curves for which $0 < \lambda < \frac{3}{2}$. On any curve on which the origin is approached in the positive quadrant (figure 2) so that $d\alpha/d\beta \to 0$,

$$a(\alpha,\beta) = a_0 + o(1), \tag{10}$$

by Lemma 1. On any steep curve, therefore, from (8) and (9),

$$\lim_{\beta\to+0} t(\alpha,\beta) = 0, \qquad \lim_{\beta\to+0} x(\alpha,\beta) = 0.$$

On any main curve, however,

$$\lim_{\beta\to+0} t(\alpha,\beta) = \mu a_0, \tag{11}$$

by (8), and then from (9),

$$\lim_{\beta\to+0} x(\alpha,\beta) = \mu a_0 u_0 - \tfrac{1}{2}\gamma\mu^2 a_0^2. \tag{12}$$

And on any flat curve, the limits of x and t do not exist: for $\lambda = 1$ and any fixed μ, in particular, Lemma 1 can be used to show that

$$\lim_{\beta\to+0} [\beta^{\frac{1}{2}} t(\alpha,\beta)] > 0 \quad \text{and} \quad \lim_{\beta\to+0} [\beta x(\alpha,\beta)] < 0$$

exist, and generally, $t \to +\infty$, $x \to -\infty$ on the flat curves.

The extraordinary point $\alpha = \beta = 0$ cannot therefore be thought of as a point, but is a concept definable only through the mode of approach. Each point (α, β) interior to G_0 has a single image (x, t), and any curve $\alpha(\beta)$ defines a family of such images. On the steep curves, as on those (part 1) approaching $\alpha = \beta = 0$ between the bore path B and the limiting characteristic L (figure 2), the limiting (x, t)-images all coincide with $x = t = 0$. But on the main curves, the limiting (x, t)-images form a curve segment P given by (11), (12). On the flat curves, no limiting (x, t)-images exist.

The curve P is the segment $t \geqslant 0$ of the parabola

$$x = u_0 t - \tfrac{1}{2}\gamma t^2 = x_s(t),$$

and since $h = (\alpha+\beta)^2/(16g) = 0$ on it (but nowhere else in G_0), it marks successive positions of the shore.† The maximum run-up distance is $x_s(u_0/\gamma) = u_0^2/(2\gamma)$ and the corresponding run-up height is $u_0^2/(2g)$ above the undisturbed water level. From $t = u_0/\gamma$ onwards, the shore recedes. Eventually, however, the parabolic shore path P wanders out of the admissible part of the formal solution, as will be seen below.

To study the water-profile near the shore line, note from the preceding results that any line $t(\alpha, \beta) = \text{const.} > 0$ must approach the extraordinary point $\alpha = \beta = 0$ like a main curve.‡ By (2) and (5)

$$\begin{aligned}\frac{dx}{dc} &= \frac{4}{1+d\alpha/d\beta}\left[(u+c)\frac{\partial t}{\partial \beta}+(u-c)\frac{\partial t}{\partial \alpha}\frac{d\alpha}{d\beta}\right]\\ &= -8c\frac{\partial t}{\partial \alpha}\frac{d\alpha}{d\beta}\bigg/\left(1+\frac{d\alpha}{d\beta}\right)\end{aligned}$$

on a curve $t = \text{const.}$, where $\partial t/\partial\beta + (\partial t/\partial\alpha)\, d\alpha/d\beta = 0$. On a main curve $\alpha = \mu\beta^{\frac{3}{2}}$,

$$\beta^{-\frac{1}{2}} d\alpha/d\beta \to 3\mu/2 \quad \text{and} \quad \beta^{\frac{3}{2}}\partial t/\partial\alpha \to a_0,$$

by (1) and (10), so $dx/dc \to -3\mu a_0$, or by (11),

$$dc/dx \to -(3t)^{-1},$$

as $c \to 0$ on a curve $t = \text{const.} > 0$; whence by (3),

$$(x_s - x)^{-2} h \to (3t)^{-2}$$

as $(x_s - x) \to 0$ for fixed $t > 0$. (At the very tip of the run-up, of course, the water profile is modified locally by effects not described by the present model.)

The genesis and precise course of the first limit line D (figure 4) depends on details of the seaward boundary condition not here considered, but as noted in § 2, the singular line approaches the point $\alpha = \beta = 0$ along a flat curve, and the limit line thus approaches the shore path P only asymptotically, as $t \to \infty$, and the general course of D is therefore roughly as indicated in figure 4. For a determination of the (x, t)-image of the 'admissible' region G^* (§ 2), it is desirable to

† Like the straight vacuum lines of gasdynamics, P is a characteristic line of both families, and no other characteristic line meets it. In contrast to the limiting characteristic line L of § I, 3, defined as $\alpha \uparrow 0$, the characteristic line $\alpha \downarrow 0$ consists of L and P. The formula $dx/dt = u$ obtained from (5) for regular images of segments of the line $\alpha+\beta = 4c = 0$ is shown by (11), (12) to be valid also in this degenerate case.

‡ Actually, $dx/dt \to u$ as $\beta \downarrow 0$ on a curve $\alpha = \mu\beta^{\frac{3}{2}}$, so that the (x, t)-images of the main curves approach P tangentially, but there exists a family of curves $\alpha/\beta^{\frac{3}{2}} = \mu + O(\beta^{\frac{1}{2}})$ on which $t = \text{const.}$ The mapping between (α, β) and (x, t) is singular all along P.

check the course of the characteristic lines $\alpha(x,t) = \text{const.}$ and $\beta(x,t) = \text{const.}$ Note that $\partial t/\partial\alpha$ and $\partial t/\partial\beta$ are of fixed sign in G^*, but by (5), $\partial x/\partial\alpha$ and $\partial x/\partial\beta$ change sign where $(u-c)$ and $(u+c)$ do. By (2) and (11),

$$u(\mu\beta^{\frac{3}{2}},\beta)-u_0 \pm c \to -\gamma\mu a_0 \quad \text{as} \quad \beta \downarrow 0,$$

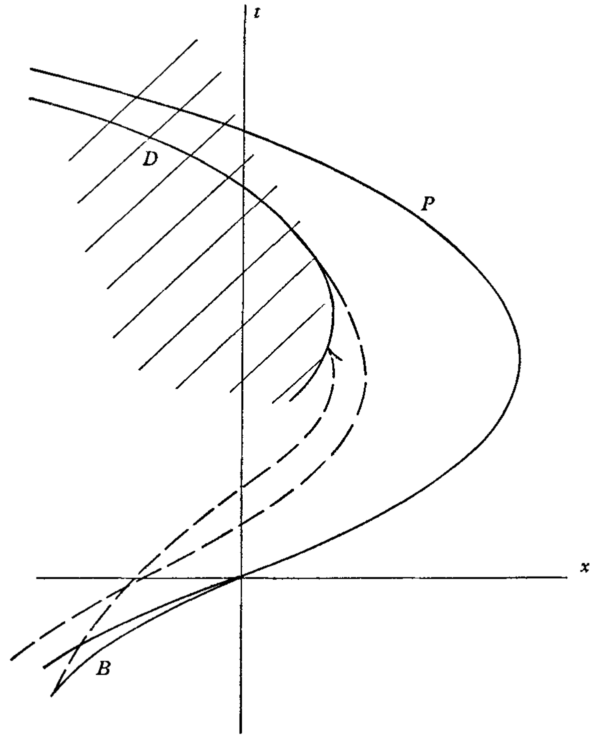

FIGURE 4. Sketch of (x, t)-diagram with bore path B, shore path P, limit line D and one characteristic line of each family (dashed).

so the lines $u \pm c = 0$ run as indicated in figure 5, with $u \pm c > 0$ to the left of these lines, since $u_0 > 0$. The 'advancing' characteristic lines $\alpha = \text{const.}$—of which the limiting characteristic L is a member (figure 1)—therefore run landward into the bore, as time increases, for $\alpha \leqslant 0$; but for $\alpha \geqslant 0$, they run landward at first and then turn seaward to approach D, which is an envelope of their family (Meyer 1949). The 'receding' characteristic lines $\beta = \text{const.}$—of which the seaward boundary C is a member (figure 1)—also run landward at first (for sufficiently small β) and then turn seaward to approach D, which is a cusp-locus of their family (Meyer 1949). The image of G^* is therefore the region between the bore path B and shore path P, on the one hand, and the shaded zone (figure 4), on the other hand.

In view of these results, it becomes possible to regard the limit line D as indicating bore formation, which is the only known physical interpretation of a limit line. Note, however, that it must be an unusual bore. On the one hand, since $a = 0$ on D, the new bore must—like the original bore B (figure 1)—be an 'advancing' bore across which the water level rises from the landward to the seaward side. On the other hand, the general course of D (figure 4) shows that the new bore—much in contrast to the original one—must be expected to move seaward, rather than landward. Of all the results noticed, this has been to us the most surprising, and it does not seem to be easily observable on our local beaches. But we have found that the secondary bore in the back-wash can indeed be observed in surf due to long swell.

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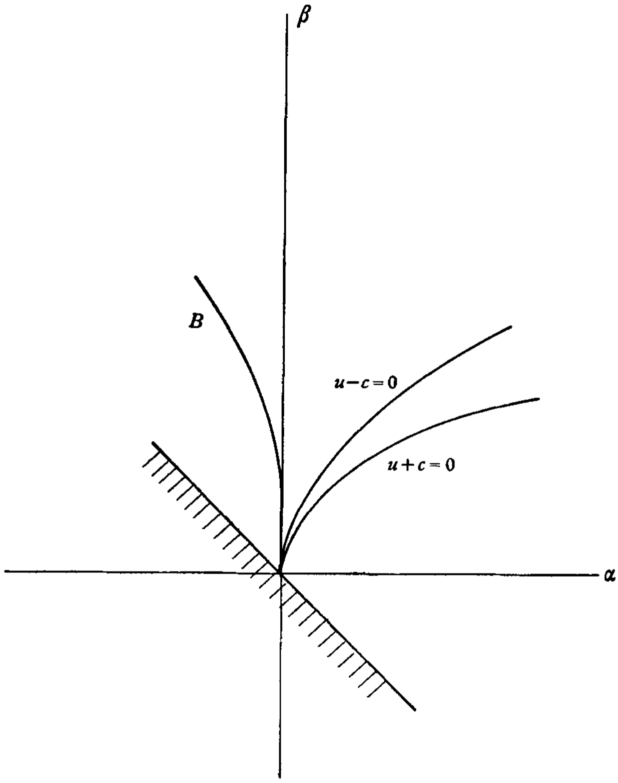

FIGURE 5. Sketch of lines $u \pm c = 0$ in the characteristic plane.

Appendix

Proof of Lemma 1

Let $Y_1 = a+b$ and $Y_2 = a-b$, then (4) and (2) imply

$$\partial^2 Y_i/\partial\alpha\,\partial\beta = n_i(\alpha+\beta)^{-2}Y_i \quad (i = 1, 2), \tag{13}$$

where $n_1 = \frac{15}{4}$, $n_2 = \frac{3}{4}$. The Riemann functions (Courant 1962) of (13) are the hypergeometric functions

$$F(a_i, b_i;\, c_i;\, z) = F_i(z),$$

$$a_1 = -\tfrac{3}{2}, \quad b_1 = \tfrac{5}{2}, \quad c_1 = 1,$$

$$a_2 = -\tfrac{1}{2}, \quad b_2 = \tfrac{3}{2}, \quad c_2 = 1,$$

and the Riemann representations of Y_i in G_δ are (Courant 1962)

$$Y(\alpha,\beta) = Y(0,\beta_0)\,F_i(\zeta_0) + \int_0^\alpha Y_{\alpha'}(\alpha',\beta_0)\,F_i(\eta)\,d\alpha' - \int_\beta^{\beta_0} Y_{\beta'}(0,\beta')\,F_i(\zeta)\,d\alpha', \tag{14}$$

if they exist, where the suffix i is omitted on Y, the noted suffix indicates partial differentiation, and

$$\eta = -(\alpha'-\alpha)(\beta_0-\beta)(\alpha'+\beta_0)^{-1}(\alpha+\beta)^{-1},$$

$$\zeta = \alpha(\beta'-\beta)\,\beta'^{-1}(\alpha+\beta)^{-1}, \qquad \zeta_0 = \alpha(\beta_0-\beta)\,\beta_0^{-1}(\alpha+\beta)^{-1}.$$

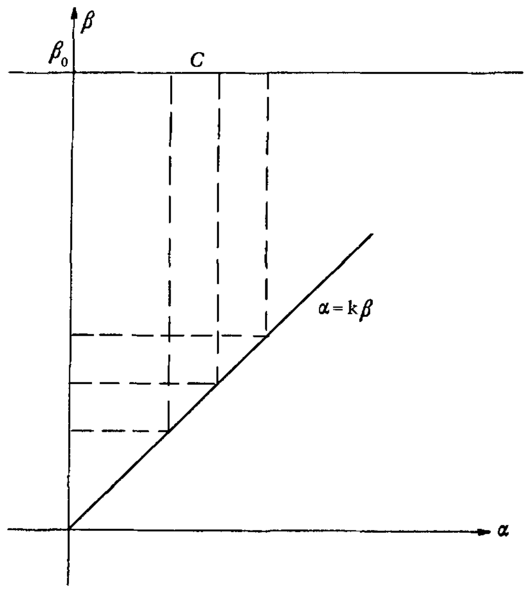

FIGURE 6. Characteristic plane with boundaries of regions corresponding to three typical members of the sequence of characteristic boundary-value problems.

We may write (14) as

$$Y(\alpha,\beta) = Y(0,\beta)\,F_i(\zeta_0) + I_1 + I_2, \tag{15}$$

$$I_1 = \int_0^\alpha Y_{\alpha'}(\alpha',\beta_0)\,F_i(\eta)\,d\alpha',$$

$$I_2 = \int_\beta^{\beta_0} Y_{\beta'}(0,\beta')\,H(\zeta;\zeta_0)\,d\beta',$$

$$H(\zeta;\,\zeta_0) = F_i(\zeta_0) - F_i(\zeta),$$

if I_1 and I_2 exist.

Now let $\alpha = k\beta$ with fixed k, always assuming $0 \leqslant k \leqslant K$. The limit $\beta \downarrow 0$ then involves a sequence of characteristic boundary-value problems determining $Y(\alpha,\beta)$ from the data on the segment $\beta \leqslant \beta' \leqslant \beta_0$ of L and the segment $0 \leqslant \alpha' \leqslant k\beta$ of C, and the latter segment will get progressively shorter, as the sequence proceeds (figure 6). Consider any member of the sequence and let β_1 denote any number such that $\beta < \beta_1 < \beta_0$ (figure 7). The second statement of (6) is as valid on $\beta = \beta_1$ as on $\beta = \beta_0$, because any discontinuities of a or $\partial a/\partial \alpha$ must persist along lines $\alpha =$ const., by (4). Since $a(0, \beta_1) > 0$ (§§I, 4, 5), it follows that the first statement of (6) remains valid for $\beta = \beta_1$, $0 \leqslant \alpha \leqslant k\beta_1$, provided β_1 is sufficiently small. For our sequence, therefore, we may replace β_0 by β_1 in (14) and (15), apply (6) on $\beta = \beta_1$, and choose β_1 sufficiently small to profit fully from (7).

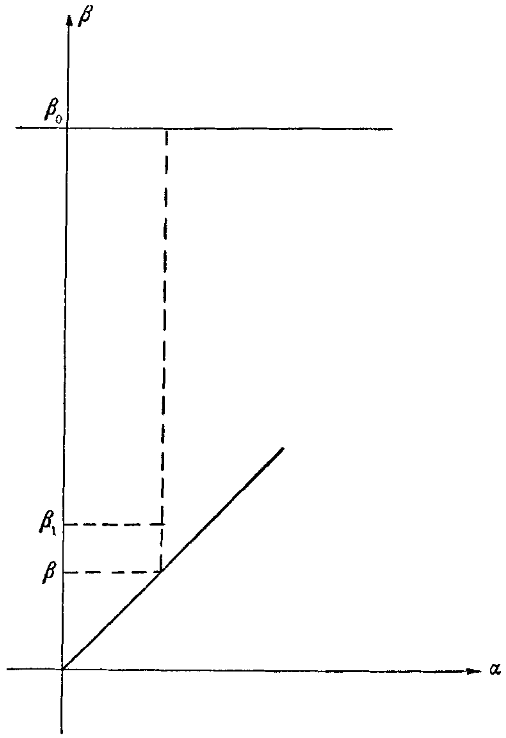

FIGURE 7. Characteristic plane showing boundary shift from β_0 to β_1.

For $0 \leqslant \alpha' \leqslant k\beta$, $0 \leqslant \beta \leqslant \beta_1$, we have

$$0 \leqslant \eta \leqslant k(\beta_1 - \beta)\beta_1^{-1}(1+k)^{-1} \leqslant K/(1+K) < 1,$$

so that $|F_i(\eta)|$ is bounded. For sufficiently small β, moreover, it follows from (6) and (4) that $Y_\alpha(\alpha, \beta_1)$ is continuous, and hence, $I_1 \to 0$ as $\beta \downarrow 0$. Next, for $\beta \leqslant \beta' \leqslant \beta_1$, we have

$$0 \leqslant (1+k)\zeta/k \leqslant (\beta_1 - \beta)/\beta_1 \leqslant 1,$$

so that $|F_i(\zeta)|$ is bounded, even if we let $\beta \downarrow 0$; and $|F_i(\zeta_0)|$ is similarly bounded. From (4) and (7), moreover, $Y_{\beta'}(0, \beta') = O(\beta'^{\frac{1}{2}})$, if β_1 be chosen sufficiently small, and hence, $I_2 \to 0$ as $\beta \downarrow 0$. But then also $\zeta_0 \to \alpha/(\alpha+\beta)$ and $Y(0, \beta) \to a_0$, by (7), and Lemma 1 follows from (15).

Remarks

An intuitive approach to Lemma 1 is offered by the observation that

$$a(0,\beta) = \text{const.} = a_0, \quad b(0,\beta) \equiv 0 \quad \text{for} \quad 0 \leqslant \beta \leqslant \beta_0 \tag{16}$$

is an obvious approximation to the boundary condition (7), and the singularity of (4) then suggests that (4) and (16) may be satisfied by functions a and b depending only on $\alpha/(\alpha+\beta)$. Their sum and difference are found from (4) to be governed by the hypergeometric equations corresponding to F_i.

Conversely, the same idea, together with the linearity of (4), may be used to deduce, from the orders noted in (7), much sharper estimates of a and b than are required in §§2, 3 above.

Lemmas 1 and 2 may be extended to prove

$$\lim_{\alpha\to+0}\lim_{\beta\to+0} a(\alpha,\beta) = -2a_0/(3\pi),$$

and for $\alpha > 0$,

$$\lim_{\beta\to+0}[b(\alpha,\beta)/\log\{\beta/(\alpha+\beta)\}] = -a_0/\pi,$$

which show the singularity of the formal solution on the positive α-axis (figure 2).

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